# EXPRESSIONS FOR CERTAIN ACCELERATIONS OF A PARTICLE.

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The object of this paper is to present a method of obtaining expressions for the magnitudes of certain accelerations of a particle in terms of its speed and from our knowledge of its path. Let the path be plane and considered only at non-singular points, and let its equation be $f(x, y) = 0$. By differentiation with respect to time, we see that the motion of the particle satisfies the equation,

$$f_x'x' + f_y'y' = 0 \tag{1}$$

where $x'$, $y'$ are time-derivatives. By differentiating (1) with respect to time and dividing by $\sqrt{f_x'^2 + f_y'^2}$, we have,

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y''y'^2}{\sqrt{f_x'^2 + f_y'^2}} + \frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = 0. \tag{2}$$

where $x''$, $y''$ are second time-derivatives. In the light of (1), we may put

$$\frac{f_x'x'' + f_y'y''}{\sqrt{f_x'^2 + f_y'^2}} = \frac{- y'x'' + x'y''}{\sqrt{x'^2 + y'^2}} = \alpha \cos \theta,$$

where $\alpha$ is the magnitude of the acceleration, and $\theta$ is the angle between the normal to the curve and the direction of acceleration. Hence for (2), we may write,

$$\frac{f_x''x'^2 + 2f_{xy}''x'y' + f_y''y'^2}{\sqrt{f_x'^2 + f_y'^2}} = - \alpha \cos \theta. \tag{3}$$

I. For example, let $f(x, y) = x^2 + y^2 - r^2$, and let the acceleration be directed toward the center of the circle. (3) now becomes the familiar formula:

$$\frac{x'^2 + y'^2}{\sqrt{x^2 + y^2}} = \alpha, \quad \text{or} \quad \alpha = \frac{v^2}{r}. \tag{4}$$

II. For another example, let $f(x, y) = b^2x^2 + a^2y^2 - a^2b^2$, and let the acceleration be directed toward one focus $F$ of the ellipse. For this example, (3) is

$$\frac{b^2x'^2 + a^2y'^2}{\sqrt{b^4x^2 + a^4y^2}} = - \alpha \cos \theta. \tag{5}$$

If $r$ and $r'$ be the focal distances from $F$ and $F'$ respectively to the particle at $P$, and $p$ and $p'$ be perpendiculars drawn from the foci to the tangent which has

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$P$ for its point of tangency, elementary properties of the ellipse are that,

$$\cos\theta = -\frac{p}{r} = -\frac{p'}{r'} = -\sqrt{\frac{pp'}{rr'}} = -\frac{b}{\sqrt{rr'}} = \frac{-b}{\sqrt{a^2 - e^2x^2}} = \frac{-ab^2}{\sqrt{b^4x^2 + a^4y^2}}.$$

And, since on the ellipse $b^2xx' + a^2yy' = 0$, the numerator in the first member of (5) has the value

$$\frac{b^4x'^2}{y^2} = \frac{a^4y'^2}{x^2}.$$

Hence (5) implies

$$(6_1)\ x'^2 = \frac{a\alpha y^2}{b^2}, \qquad (6_2)\ y'^2 = \frac{b^2\alpha x^2}{a^3};$$

whence

$$v^2 = \frac{\alpha(a^4y^2 + b^4x^2)}{a^3b^2} = \frac{\alpha rr'}{a},$$

or

$$\alpha = \frac{av^2}{rr'}, \tag{7}$$

which is a generalization of (4).

By differentiating ($6_1$) as to time, and setting in the value of $y'$ from ($6_2$), we have,

$$x'' = \frac{1}{2a}\frac{d\alpha}{dx}(a^2 - x^2) - \frac{x\alpha}{a}.$$

But the $x$-component of acceleration is

$$x'' = \alpha\frac{ae - x}{a - ex}.$$

From the comparison of the last two equations, we have

$$\frac{d\alpha}{2\alpha} = \frac{e\,dx}{a - ex}.$$

Whence the Newtonian law:

$$\alpha = \frac{k^2}{(a - ex)^2} = \frac{k^2}{r^2}. \tag{8}$$

A comparison of (7) and (8) yields the following elegant form of the *vis viva* equation:

$$v^2 = \frac{k^2}{a}\,\frac{r'}{r}.$$

The above discussion of elliptical motion, with only the slightest modification is valid for hyperbolic and parabolic motion under a central force directed toward a focus. And the last equation is valid in all three cases without modification, $r'/a$ having the limiting value 2 in the case of the parabola.

III. For the path of a projectile, we may take a parabola of higher order and let

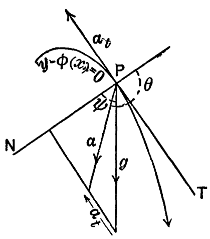
$$f(x, y) = y - (a_0 + a_1x + a_2x^2 + \dots + a_nx^n) = y - \varphi(x).$$

The range is assumed so short that the vertical direction may be regarded as constant. Now $\alpha$ is the resultant of the vertical gravitational acceleration $-\ g(y)$, and a tangential acceleration $\alpha_t$. Obviously $\alpha \cos \theta = -\ g \cos \psi$, where $\psi$ is the angle between the normal and the vertical. Hence (3) for this example is, by reference to (1),

$$-\frac{\varphi''(x)x'^2}{\sqrt{f_x'^2 + f_y'^2}} = g \cos \psi = g\frac{dx}{\sqrt{dx^2 + dy^2}} = g\frac{f_y'}{\sqrt{f_x'^2 + f_y'^2}}.$$

or,

$$x' = \frac{\sqrt{g}}{[-\ \varphi''(x)]^{1/2}}.$$




Therefore,

(9)
$$v = \frac{\sqrt{g}}{[-\ \varphi''(x)]^{1/2}}\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

and,

$$x'' = \frac{g}{2}\frac{\varphi'''(x)}{[\varphi''(x)]^2}.$$

But since the total horizontal acceleration is the horizontal component of $\alpha_t$, it follows that

(10)
$$\alpha_t = \frac{g\varphi'''(x)}{2[\varphi''(x)]^2}\cdot\sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

or, by (9),

$$\alpha_t = v\frac{\sqrt{g}\varphi'''(x)}{2[-\ \varphi''(x)]^{3/2}}.$$

Now (9) and (10) can be written in the form $v = F_1(x)$ and $\alpha_t = F_2(x)$, where $F_1$ and $F_2$ are power series. The method of inversion of series gives

(11)
$$F_2^{-1}(\alpha_t) = F_1^{-1}(v).$$

This entire discussion of the motion of a projectile is valid if the projectile is a finite sphere and also if the medium is not homogeneous. But the relation of $\alpha_t$ to $v$ depends on the character and condition of the medium. So, (11) has the nature of a law for the particular projectile if, and only if, the medium in which the flight occurred was equally resistant at all points in the observed path.